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EFFECTS OF CHANGE OF REFERENCE COORDINATES ON THE STRESS ANALYSES OF ANISOTROPIC ELASTIC MATERIALS

T. C. T. TING

Department of Materials Engineering, University of Illinois at Chicago Circle, Chicago, IL 60680, U.S.A.

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Abstract—When the displacements are independent of the x_3-coordinate, the eigenvalues and eigenvectors of the anisotropic elasticity constants depend on the orientation of the (x_1, x_2) axes. It is shown that the components of the eigenvectors (which are complex values) transform according to the law of transformation for tensors of order one. The transformation of the eigenvalues is more complicated. The effects of change of the reference coordinates on the form of general solution are discussed. Also discussed is the form of general solution when the eigenvalue p is a multiple root. Finally, we show that as the angle of rotation ϕ of the coordinate axes varies from 0 to 2π, each p traverses a circle in the complex plane which is orthogonal to the unit circle with center at the origin. A graphical solution of the eigenvalue p for a given ϕ is presented. Some functions of p which are invariant to the rotation of the coordinate axes are obtained.

1. INTRODUCTION

The general solution to the class of problems in anisotropic elastic materials in which the displacement and hence the stress is independent of the x_3-coordinate in a rectangular coordinate system was first obtained by Eshelby *et al.*[1]. They applied their results to a straight dislocation acted on by a concentrated body force. This was extended to a line singularity by Stroh[2] who subsequently developed a powerful six-dimensional theory of dislocations and surface waves in anisotropic solids[3]. Stroh's theory has been further developed by Barnett and his co-workers in a series of papers (see[4–6], e.g.). An excellent review article on the theory of surface waves in anisotropic elastic materials was given recently by Chadwick and Smith[7].

Basic to the analyses of anisotropic materials is the eigenvalues p and the associated eigenvectors $\mathbf{g}$ and $\mathbf{h}$, called the Stroh eigenvectors of the elasticity constants. Since the elasticity constants depend on the choice of the reference coordinates, so do the eigenvalues and the Stroh eigenvectors. After presenting the basic equations necessary for the paper in Section 2, we study in Section 3 how $\mathbf{g}$, $\mathbf{h}$ and p vary as one rotates the coordinate system about the x_3-axis. We find that $\mathbf{g}$ and $\mathbf{h}$ transform according to the law of transformation for tensors of order one. This result may appear to be in contradiction with that of Barnett and Lothe[5] and we explain the differences in the interpretation of the results in Section 4. In Section 5 we investigate how the form of general solution changes due to the change of the reference coordinates. This is relevant to the analyses of stress singularities at the vertex of an anisotropic composite wedge[8] because the order of singularities should be independent of the choice of the reference coordinates. The form of the general solution becomes incomplete when there is at least one multiple eigenvalue. We present in Section 6 a general solution associated with a multiple eigenvalue. Finally, we investigate geometrically in Section 7 how each of the eigenvalues p varies as the angle ϕ of the rotation of the coordinate axes varies from 0 to 2π. We show that the locus of p in the complex plane is a circle as ϕ varies from 0 to 2π. The circle intersects orthogonally another circle of unit radius with center at the origin. A graphical solution of p for a given ϕ is presented. We also present some functions of p which are invariant to the rotation of the coordinate axes.

2. BASIC EQUATIONS

We use rectangular cartesian coordinates (x_1, x_2, x_3) and adopt the convention of implied summation over repeated subscript indices from 1 to 3. The constitutive and equilibrium equations are

$$\sigma_{ij} = c_{ijkl} u_{k,l} \tag{1}$$

$$\sigma_{ij,j} = 0 \tag{2}$$

where u_i and σ_{ij} are the displacement and stress, respectively, and a comma stands for partial derivative with respect to the space coordinates. c_{ijkl} are the material constants with the symmetry properties

$$c_{ijkl} = c_{klij} = c_{jikl} \tag{3}$$

We assume that u_i and σ_{ij} are independent of x_3. Equation (2) then reduces to

$$\sigma_{i1,1} + \sigma_{i2,2} = 0. \tag{4}$$

It can be shown that[7] the vector s defined by

$$s_i(x_1, x_2) = \int_{a_1}^{x_1} \sigma_{i2}(\xi, a_2)\, d\xi - \int_{a_2}^{x_2} \sigma_{i1}(x_1, \eta)\, d\eta \tag{5}$$

where a_1 and a_2 are arbitrary real constants, generates the stress components

$$\sigma_{i1} = -s_{i,2}, \quad \sigma_{i2} = s_{i,1} \tag{6}$$

and hence eqn (4) is automatically satisfied. Elimination of σ_{ij} between eqns (1) and (6) yields the relations

$$\left.\begin{aligned} Q_{ik} u_{k,1} + R_{ik} u_{k,2} &= -s_{i,2} \\ R_{ki} u_{k,1} + T_{ik} u_{k,2} &= s_{i,1} \end{aligned}\right\} \tag{7}$$

where

$$\left.\begin{aligned} Q_{ik} &= c_{i1k1} \\ R_{ik} &= c_{i1k2} \\ T_{ik} &= c_{i2k2}. \end{aligned}\right\} \tag{8}$$

Equations (7) can be written in matrix notations as

$$\begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{R}^T & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{,1} \\ \mathbf{s}_{,1} \end{bmatrix} + \begin{bmatrix} \mathbf{R} & \mathbf{I} \\ \mathbf{T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{,2} \\ \mathbf{s}_{,2} \end{bmatrix} = \mathbf{0} \tag{9}$$

where the superscript T stands for the transpose.

Introducing the new variable Z by

$$Z = x_1 + p x_2 \tag{10}$$

where p is a constant, the general solution for **u** and **s** may be written as

$$u_i = g_i f(Z) + \ldots \tag{11a}$$

$$s_i = h_i f(Z) + \ldots \tag{11b}$$

where f is an arbitrary function and **g** and **h** are the eigenvectors. Since there are six eigenvalues p and six associated eigenvectors $\mathbf{g}$ and $\mathbf{h}$, the dots in eqns (11) stand for the remaining five arbitrary functions. For simplicity only the first term is written explicitly in eqn (11) to avoid introducing an additional subscript for the eigenvalues. $\mathbf{g}$, $\mathbf{h}$ and p are determined by substituting eqns (11) into (9):

$$\begin{bmatrix} \mathbf{Q}+p\mathbf{R} & p\mathbf{I} \\ \mathbf{R}^T+p\mathbf{T} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{h} \end{bmatrix} = \mathbf{0} \tag{12}$$

or

$$p\mathbf{h} = -(\mathbf{Q}+p\mathbf{R})\mathbf{g} \tag{13a}$$

$$\mathbf{h} = (\mathbf{R}^T + p\mathbf{T})\mathbf{g}. \tag{13b}$$

Elimination of $\mathbf{h}$ between the two equations yields

$$\mathbf{D}\mathbf{g} = \mathbf{0}, \tag{14}$$

where

$$\mathbf{D} = \mathbf{Q} + p(\mathbf{R}+\mathbf{R}^T) + p^2\mathbf{T}. \tag{15}$$

For the nontrivial solution of $\mathbf{g}$, the determinant of $\mathbf{D}$ must vanish. This provides a sextic equation for p. If the strain energy is positive definite, it can be shown that p cannot be real[1, 3, 9]. Therefore, we have three pairs of complex conjugate roots for p. The eigenvectors $\mathbf{g}$ and $\mathbf{h}$ are called the Stroh eigenvectors and are obtained from eqns (14) and (13b).

Before we go to the next section, we write σ_{ij} in terms of the arbitrary function $f(Z)$ using eqns (6) and (11b):

$$\left.\begin{aligned} \sigma_{i1} &= -ph_i \frac{\mathrm{d}f(Z)}{\mathrm{d}Z} - \dots \\ \sigma_{i2} &= h_i \frac{\mathrm{d}f(Z)}{\mathrm{d}Z} + \dots \end{aligned}\right\} \tag{16a}$$

The only stress component missing in eqn (16a) is σ_{33} which is obtained from eqns (1) and (11a):

$$\sigma_{33} = (c_{33k1} + pc_{33k2})\, g_k \frac{\mathrm{d}f(Z)}{\mathrm{d}Z} + \dots \tag{16b}$$

Since the change of (x_1, x_2) axes does not affect σ_{33}, we will ignore σ_{33} in the rest of the paper.

3. CHANGE OF REFERENCE COORDINATES

Consider a new reference coordinate x_i^* which is obtained by rotating the x_i coordinates about the x_3-axis an angle ϕ, Fig. 1. Hence

$$\left.\begin{aligned} x_1^* &= x_1 \cos\phi + x_2 \sin\phi \\ x_2^* &= -x_1 \sin\phi + x_2 \cos\phi \\ x_3^* &= x_3 \end{aligned}\right\} \tag{17}$$

Let

$$\Omega_{ij} = \frac{\partial x_i^*}{\partial x_j} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

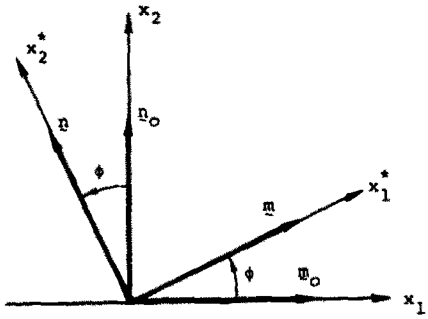

Fig. 1. Change of reference coordinates.

Since c_{ijkl} is a tensor of order 4, its components c^*_{ijkl} in the x^*_i coordinates are related to c_{ijkl} by the relation

$$c^*_{ijkl} = \Omega_{ip}\Omega_{jq}\Omega_{kr}\Omega_{ls}c_{pqrs}. \tag{19}$$

Noticing that

$$\left.\begin{aligned}\Omega_{1j} &= \delta_{j1}\cos\phi + \delta_{j2}\sin\phi \\ \Omega_{2j} &= -\delta_{j1}\sin\phi + \delta_{j2}\cos\phi\end{aligned}\right\} \tag{20}$$

where δ_{ij} is the Kronecker delta, it can be shown from eqns (8), (19) and (20) that

$$\mathbf{Q}^* = c^*_{i1k1} = \mathbf{\Omega}\{\mathbf{Q}\cos^2\phi + \mathbf{T}\sin^2\phi + (\mathbf{R}+\mathbf{R}^T)\cos\phi\sin\phi\}\,\mathbf{\Omega}^T \tag{21a}$$

$$\mathbf{R}^* = c^*_{i1k2} = \mathbf{\Omega}\{\mathbf{R}\cos^2\phi - \mathbf{R}^T\sin^2\phi + (\mathbf{T}-\mathbf{Q})\cos\phi\sin\phi\}\,\mathbf{\Omega}^T \tag{21b}$$

$$\mathbf{T}^* = c^*_{i2k2} = \mathbf{\Omega}\{\mathbf{T}\cos^2\phi + \mathbf{Q}\sin^2\phi - (\mathbf{R}+\mathbf{R}^T)\cos\phi\sin\phi\}\,\mathbf{\Omega}^T \tag{21c}$$

Rewriting eqn (15) for the x_i^* coordinates as

$$\mathbf{D}^* = \mathbf{Q}^* + p^*(\mathbf{R}^* + \mathbf{R}^{*T}) + p^{*2}\mathbf{T}^* \tag{22}$$

and substituting eqns (21) into (22) yields

$$\mathbf{D}^* = (\cos\phi - p^*\sin\phi)^2\,\mathbf{\Omega}\mathbf{D}\mathbf{\Omega}^T \tag{23}$$

where p, which is contained in $\mathbf{D}$ above, has been set to

$$p = (\sin\phi + p^*\cos\phi)/(\cos\phi - p^*\sin\phi). \tag{24}$$

Solving for p^*, we have

$$\begin{aligned}p^* &= (p\cos\phi - \sin\phi)/(p\sin\phi + \cos\phi) \\ &= \tan(\psi - \phi)\end{aligned} \tag{25}$$

where ψ is a complex angle. Both expressions for p^* in eqn (25) have been obtained before[9,5] by different approaches.

The eigenvectors $\mathbf{g}^*$ and $\mathbf{h}^*$ in the x^*_i coordinates are obtained by rewriting eqns (14) and

(13b) as

$$\mathbf{D}^*\mathbf{g}^* = \mathbf{0} \tag{26}$$

$$\mathbf{h}^* = (\mathbf{R}^{*T} + p^*\mathbf{T}^*)\mathbf{g}^*. \tag{27}$$

Noticing that $(\cos\phi - p^*\sin\phi) \neq 0$ because p^* is a complex number, eqn (26) gives, using eqns (23) and (14), $\mathbf{\Omega}^T\mathbf{g}^* = \lambda\mathbf{g}$ where λ is a constant. Upon normalization, we have

$$\mathbf{g}^* = \mathbf{\Omega}\mathbf{g}. \tag{28a}$$

Substitution of eqns (21), (25) and (28a) into eqn (27) and making use of eqns (13) yields

$$\mathbf{h}^* = \mathbf{\Omega}\mathbf{h}. \tag{28b}$$

Equations (28a,b) show that the Stroh eigenvectors **g** and **h** are *frame-indifference* vectors according to the definition of Truesdell and Knoll[10]. This would have been expected had the components of **g** and **h** been real values. Nevertheless, we may regard vectors **g** and **h** as having fixed directions in the material. Therefore, they should not change their directions with change of the reference coordinates. The *components* of **g** and **h**, of course, change according to eqns (28) when the frame of reference changes.

4. COMPARISON WITH THE PUBLISHED RESULTS

Let **m** and **n** be unit vectors along the x_1^* – and x_2^* – coordinate axes, respectively, Fig. 1. Then

$$\left.\begin{aligned} \mathbf{m} &= (\cos\phi, \sin\phi, 0) \\ \mathbf{n} &= (-\sin\phi, \cos\phi, 0) \\ \mathbf{m}_0 &= (1,0,0), \quad \mathbf{n}_0 = (0,1,0). \end{aligned}\right\} \tag{29}$$

Instead of assuming **u** and **s** in the form given in eqns (11), Barnett and Lothe[6] considered a more general form:

$$u_i = \hat{g}_i f^*(\mathbf{m}\cdot\mathbf{x} + p^*\mathbf{n}\cdot\mathbf{x}) + \ldots \tag{30a}$$

$$s_i = \hat{h}_i f^*(\mathbf{m}\cdot\mathbf{x} + p^*\mathbf{n}\cdot\mathbf{x}) + \ldots \tag{30b}$$

Since **m**, **n** and p^* depend on ϕ, one might expect that $\hat{g}_i$ and $\hat{h}_i$ also depend on ϕ. Barnett and Lothe obtained the remarkable result that $\hat{g}_i$ and $\hat{h}_i$ are independent of ϕ. Noting that the argument of f^* in eqns (30) reduces to Z when $\phi = 0$, this implies that $\hat{g}_i$ and $\hat{h}_i$ are identical to g_i and h_i, respectively. Before we prove the invariance of $\hat{g}_i$ and $\hat{h}_i$ with respect to ϕ using the approaches of this paper, it should be pointed out that u_i and s_i *in eqns* (30) *are vector components referred to the fixed reference frame* x_i, not the rotating frame x_i^*. *Therefore,* $\hat{g}_i$ *and* $\hat{h}_i$ *should not be regarded as the Stroh eigenvectors in the* x_i^* *coordinates.*

Following[6], we define

$$(\mathbf{mn})_{ik} = c_{ijkl}m_j n_l. \tag{31}$$

Substitution of eqn (30a) into eqn (1) and then into eqn (2) leads to the relation

$$\hat{\mathbf{D}}\hat{\mathbf{g}} = \mathbf{0} \tag{32}$$

where

$$\hat{\mathbf{D}} = (\mathbf{mm}) + p^*(\mathbf{mn} + \mathbf{nm}) + p^{*2}(\mathbf{nn}) \tag{33}$$

Using eqns (29) and (31), it can be shown that (see also p. 313 of [7])

$$\left.\begin{aligned}(\mathbf{mm}) &= \mathbf{Q}\cos^2\phi + \mathbf{T}\sin^2\phi + (\mathbf{R}+\mathbf{R}^T)\cos\phi\sin\phi\\ (\mathbf{mn}) &= \mathbf{R}\cos^2\phi - \mathbf{R}^T\sin^2\phi + (\mathbf{T}-\mathbf{Q})\cos\phi\sin\phi\\ (\mathbf{nn}) &= \mathbf{T}\cos^2\phi + Q\sin^2\phi - (\mathbf{R}+\mathbf{R}^T)\cos\phi\sin\phi\end{aligned}\right\} \tag{34}$$

or, in view of eqns (21),

$$\left.\begin{aligned}(\mathbf{mm}) &= \mathbf{\Omega}^T\mathbf{Q}^*\mathbf{\Omega}\\ (\mathbf{mn}) &= \mathbf{\Omega}^T\mathbf{R}^*\mathbf{\Omega}\\ (\mathbf{nn}) &= \mathbf{\Omega}^T\mathbf{T}^*\mathbf{\Omega}.\end{aligned}\right\} \tag{35}$$

It follows from eqns (33), (35), (22) and (23) that

$$\hat{\mathbf{D}} = (\cos\phi - p^*\sin\phi)^2\,\mathbf{D} = (\cos\phi + p\sin\phi)^{-2}\,\mathbf{D}. \tag{36}$$

Equations (32), (36) and (14) indicate that, using the same argument in deriving eqn (28a),

$$\hat{\mathbf{g}} = \mathbf{g} \tag{37}$$

and hence $\hat{\mathbf{g}}$ is an invariant.

If we eliminate σ_{i2} between eqns (1) and (6) and make use of eqns (30), we obtain a relation between $\hat{\mathbf{g}}$ and $\hat{\mathbf{h}}$ which can be simplified after using eqns (29) and (24):

$$\hat{\mathbf{h}} = (\mathbf{R}^T + p\mathbf{T})\hat{\mathbf{g}} = \mathbf{h}. \tag{38}$$

The last equality comes from eqns (37) and (13b). Hence $\hat{\mathbf{h}}$ is an invariant.

Before we go the the next section, we obtain from eqns (11a), (30a) and (37) that

$$f^*(\mathbf{m}\cdot\mathbf{x} + p^*\mathbf{n}\cdot\mathbf{x}) = f(Z). \tag{39}$$

Since the r.h.s. is independent of ϕ, so is the l.h.s. Therefore, $f^*(\mathbf{m}\cdot\mathbf{x} + p^*\mathbf{n}\cdot\mathbf{x})$ *is an invariant with respect to* ϕ. In other words, $f^*(\mathbf{m}\cdot\mathbf{x} + p^*\mathbf{n}\cdot\mathbf{x})$ is a constant for a fixed $\mathbf{x}$.

5. TRANSFORMATION OF THE GENERAL SOLUTION

In [8], the stress singularities at the vertex of a wedge or a composite wedge of anisotropic materials were considered by assuming the form of f in eqns (11) in the following form:

$$f(Z) = Z^{1-\kappa}/(1-\kappa) \tag{40}$$

where κ is a constant. The origin $x_1 = x_2 = 0$ is the vertex. Since $Z = x_1 + px_2$ and there are three pairs of complex conjugates for the eigenvalues, eqns (11) have the form:

$$u_i = (A_1 g_i Z^{1-\kappa} + B_1 \bar{g}_i \bar{Z}^{1-\kappa})/(1-\kappa) + \ldots \tag{41a}$$

$$s_i = (A_1 h_i Z^{1-\kappa} + B_1 \bar{h}_i \bar{Z}^{1-\kappa})/(1-\kappa) + \ldots \tag{41b}$$

where A_1, $B_1, \ldots$ are constants which are in general complex and an overbar denotes a complex conjugate. For simplicity only the terms associated with one pair of eigenvalues are written explicitly to avoid introducing an additional subscript for the eigenvalues and Z. The dots in eqns (41) denote terms associated with the remaining two pairs of eigenvalues. It follows from eqns (6) and (41b) that the stresses σ_{ij} are given by

$$\sigma_{i1} = -(A_1 p h_i Z^{-\kappa} + B_1 \bar{p}\bar{h}_i \bar{Z}^{-\kappa}) - \dots \quad (42a)$$

$$\sigma_{i2} = (A_1 h_i Z^{-\kappa} + B_1 \bar{h}_i \bar{Z}^{-\kappa}) + \dots \quad (42b)$$

For a single wedge, substitution of eqns (41a) and (42) into the boundary conditions results in a system of linear algebraic equations for A_1, $B_1, \dots$, which may be written as

$$\mathbf{K}\mathbf{a} = \mathbf{b} \quad (43)$$

where **K** is a square matrix which depends on κ, **a** is a column matrix whose elements are A_1, $B_1, \dots$, and **b** is a column matrix which depends on the boundary conditions.

For a composite wedge, writing equations similar to eqns (41) and (42) for each material and substituting the resulting equations in the boundary and interface conditions, one obtains a system of linear equations in A_1, $B_1, \dots$, which again can be written in the form of eqn (43).

When the boundary and interface conditions are homogeneous, $\mathbf{b} \equiv \mathbf{0}$ and a nontrivial solution for **a** exists if the determinant of **K** vanishes. This provides the constant κ. If the real part of κ is positive, we see from eqns (42) that the stress is singular at the vertex.

The following question† arises: Since the order of singularity is independent of the choice of the reference coordinates, how does one see this from the formulation presented in eqns (40) through (43)?

A similar question could be asked of other boundary value problems. We will therefore use the general solution given by eqns (11) and consider the effects of change of reference coordinates on the arbitrary function $f(Z)$.

In the x_i^*-coordinates, Z, u_i, s_i and σ_{ij} are rewritten as

$$Z^* = x_1^* + p^* x_2^* = \mathbf{m} \cdot \mathbf{x} + p^* \mathbf{n} \cdot \mathbf{x} \quad (44)$$

$$\left.\begin{aligned} u_i^* &= g_i^* f^*(Z^*) + \dots \\ s_i^* &= h_i^* f^*(Z^*) + \dots \end{aligned}\right\} \quad (45)$$

$$\left.\begin{aligned} \sigma_{i1}^* &= -p^* h_i^* \frac{df^*(Z^*)}{dZ^*} - \dots \\ \sigma_{i2}^* &= h_i^* \frac{df^*(Z^*)}{dZ^*} + \dots \end{aligned}\right\} \quad (46)$$

Using eqns (17) and (25), it can be shown that

$$Z^* = Z/\zeta = (\mathbf{m}_0 \cdot \mathbf{x} + p\,\mathbf{n}_0 \cdot \mathbf{x})/\zeta \quad (47)$$

where

$$\zeta = \cos\phi + p \sin\phi = \mathbf{m}_0 \cdot \mathbf{m} + p\,\mathbf{n}_0 \cdot \mathbf{m}. \quad (48)$$

Equation (39) can be written as, using eqn (47),

$$f^*(Z^*) = f(Z) = f(\zeta Z^*) \quad (49)$$

Hence

$$\frac{df^*(Z^*)}{dZ^*} = \frac{df(Z)}{dZ}\zeta. \quad (50)$$

†See the Acknowledgements at the end of the paper.

Equations (45) and (46) reduce to, using eqns (28), (49), (50), (25) and (11),

$$\left.\begin{aligned} u_i^* &= \Omega_{ik} g_k f(Z) = \Omega_{ik} u_k \\ s_i^* &= \Omega_{ik} h_k f(Z) = \Omega_{ik} s_k \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} \sigma_{i1}^* &= -(p\cos\phi - \sin\phi)\,\Omega_{ik} h_k \frac{\mathrm{d}f(Z)}{\mathrm{d}Z} - \ldots \\ \sigma_{i2}^* &= (p\sin\phi + \cos\phi)\Omega_{ik} h_k \frac{\mathrm{d}f(Z)}{\mathrm{d}Z} + \ldots \end{aligned}\right\} \tag{52}$$

Equations (52) can be rewritten as, using eqns (16a) and (20)

$$\left.\begin{aligned} \sigma_{i1}^* &= \Omega_{ik}(\sigma_{k1}\cos\phi + \sigma_{k2}\sin\phi) \\ &= \Omega_{ik}\Omega_{1q}\sigma_{kq} \end{aligned}\right\} \tag{53a}$$

$$\left.\begin{aligned} \sigma_{i2}^* &= \Omega_{ik}(-\sigma_{k1}\sin\phi + \sigma_{k2}\cos\phi) \\ &= \Omega_{ik}\Omega_{2q}\sigma_{kq}. \end{aligned}\right\} \tag{53b}$$

As expected, Eqns (51) and (53) show that **u** and **s** transform as tensors of order one while σ_{ij} transform as tensors of order two.

The main results in this section are eqns (47)–(53). These equations enable one to see how a solution in the x_i coordinates is related to the solution in the x_i^* coordinates.

6. DEGENERACY OF THE EIGENVALUES

The analyses presented so far tacitly assume that the eigenvalues p's are distinct. When one of p's is a double root, one may or may not have two independent solutions for the eigenvectors **g** and **h**,[7]. When there is only one independent solution for the eigenvector associated with the double root eigenvalue, the general solution given by eqns (11) will not provide six independent arbitrary functions. In[8] the second independent solution was derived for the function $f(Z)$ given in the form of eqn (40). We will derive in this section the second independent solution for arbitrary $f(Z)$. The case in which p is a triple root will also be discussed.

When p is a double root, the first independent solution is given by the first term in eqns (11) while the second independent solution is

$$u_i = \frac{\mathrm{d}}{\mathrm{d}p}\{g_i f(Z)\} = \frac{\mathrm{d}g_i}{\mathrm{d}p} f(Z) + g_i \frac{\mathrm{d}f}{\mathrm{d}Z} x_2 \tag{54a}$$

$$s_i = \frac{\mathrm{d}}{\mathrm{d}p}\{h_i f(Z)\} = \frac{\mathrm{d}h_i}{\mathrm{d}p} f(Z) + h_i \frac{\mathrm{d}f}{\mathrm{d}Z} x_2 \tag{54b}$$

where $\mathrm{d}g_i/\mathrm{d}p$ and $\mathrm{d}h_i/\mathrm{d}p$ are obtained by differentiating eqns (14) and (13b):

$$\mathbf{D}\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}p} + \frac{\mathrm{d}\mathbf{D}}{\mathrm{d}p}\mathbf{g} = \mathbf{0} \tag{55}$$

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}p} = (\mathbf{R}^T + p\mathbf{T})\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}p} + \mathbf{T}\mathbf{g}. \tag{56}$$

We will not discuss here the existence of a solution for **g** and $\mathrm{d}\mathbf{g}/\mathrm{d}p$ from eqns (14) and (55) (see [8, 11]). Using the relation,

$$x_2 = (Z - \bar{Z})/(p - \bar{p}) = (Z - \bar{Z})/(2\beta i), \tag{57}$$

where β is the imaginary part of p, and deleting the term $g_i Z\, \mathrm{d}f(Z)/\mathrm{d}Z$ and $h_i Z\, \mathrm{d}f(Z)/\mathrm{d}Z$ which can be absorbed in the first independent solution, one has

$$u_i = \frac{\mathrm{d}g_i}{\mathrm{d}p} f(Z) - \frac{g_i}{2\beta i}\frac{\mathrm{d}f(Z)}{\mathrm{d}Z}\bar{Z} \tag{58a}$$

$$s_i = \frac{\mathrm{d}h_i}{\mathrm{d}p} f(Z) - \frac{h_i}{2\beta i}\frac{\mathrm{d}f(Z)}{\mathrm{d}Z}\bar{Z}. \tag{58b}$$

From eqn (6), the stresses obtained from s_i are

$$\sigma_{i1} = -\left(p\frac{\mathrm{d}h_i}{\mathrm{d}p} - \bar{p}\frac{h_i}{2\beta i}\right)\frac{\mathrm{d}f(Z)}{\mathrm{d}Z} + \frac{p h_i}{2\beta i}\frac{\mathrm{d}^2 f(Z)}{\mathrm{d}Z^2}\bar{Z} \tag{59a}$$

$$\sigma_{i2} = \left(\frac{\mathrm{d}h_i}{\mathrm{d}p} - \frac{h_i}{2\beta i}\right)\frac{\mathrm{d}f(Z)}{\mathrm{d}Z} - \frac{h_i}{2\beta i}\frac{\mathrm{d}^2 f(Z)}{\mathrm{d}Z^2}\bar{Z}. \tag{59b}$$

We can obtain a little simpler expression for σ_{ij} if we choose, instead of eqn (58), the following solution:

$$u_i = \left(\frac{\mathrm{d}g_i}{\mathrm{d}p} + \frac{g_i}{2\beta i}\right) f(Z) - \frac{g_i}{2\beta i}\frac{\mathrm{d}f(Z)}{\mathrm{d}Z}\bar{Z} \tag{60a}$$

$$s_i = \left(\frac{\mathrm{d}h_i}{\mathrm{d}p} + \frac{h_i}{2\beta i}\right) f(Z) - \frac{h_i}{2\beta i}\frac{\mathrm{d}f(Z)}{\mathrm{d}Z}\bar{Z}. \tag{60b}$$

We then have

$$\sigma_{i1} = -\left(p\frac{\mathrm{d}h_i}{\mathrm{d}p} + h_i\right)\frac{\mathrm{d}f(Z)}{\mathrm{d}Z} + \frac{p h_i}{2\beta i}\frac{\mathrm{d}^2 f(Z)}{\mathrm{d}Z^2}\bar{Z} \tag{61a}$$

$$\sigma_{i2} = \frac{\mathrm{d}h_i}{\mathrm{d}p}\frac{\mathrm{d}f(Z)}{\mathrm{d}Z} - \frac{h_i}{2\beta i}\frac{\mathrm{d}^2 f(Z)}{\mathrm{d}Z^2}\bar{Z}. \tag{61b}$$

If p is a triple root, a third independent solution is

$$u_i = \frac{\mathrm{d}^2}{\mathrm{d}p^2}\{g_i f(Z)\} \tag{62a}$$

$$s_i = \frac{\mathrm{d}^2}{\mathrm{d}p^2}\{h_i f(Z)\}. \tag{62b}$$

We will not discuss this solution further since we have not seen an example other than isotropic materials for which p is a triple root. For isotropic materials, the displacement u_3 is uncoupled from u_1 and u_2. Therefore, the eigenvalue p is a double root with respect to the plane deformations with displacements u_1 and u_2 and the solution, eqns (58) or (60), applies. For the anti plane deformation which involves u_3 alone, p is a single root [8].

7. PROPERTIES OF THE EIGENVALUES

The eigenvalue p, which is a root of the determinant of $\mathbf{D}$ defined in eqn (15), plays important roles in the analyses of composite materials [12]. For isotropic materials, $p = \pm i$ is a triple root and is independent of the choice of reference coordinates. For anisotropic materials, p depends on the choice of the reference coordinates. It was stated in[5] that, in the complex p-plane, p encircles the point $+i$ or $-i$ twice as the angle ϕ of Fig. 1 varies from 0 to 2π. In this section we will look at more closely the precise locus of p as ϕ varies from 0 to 2π.

We first write eqn (25) in the following form:

$$p(\phi)=\frac{p(0)\cos\phi-\sin\phi}{p(0)\sin\phi+\cos\phi}. \tag{63}$$

Let α and β be, respectively, the real and imaginary parts of $p(\phi)$:

$$p(\phi)=\alpha(\phi)+i\beta(\phi) \tag{64}$$

$$p(0)=\alpha_o+i\beta_0. \tag{65}$$

Equation (63) can be rewritten in the form:

$$\alpha(\phi)+i\beta(\phi)=\frac{\alpha_0\cos 2\phi+\frac{1}{2}(\alpha_0{}^2+\beta_0{}^2-1)\sin 2\phi+i\beta_0}{(\alpha_0\sin\phi+\cos\phi)^2+(\beta_0\sin\phi)^2}. \tag{66}$$

It follows from eqn (66) that *if* $p(0)=\pm i$, $p(\phi)=\pm i$ *for all* ϕ. *Also, if p is a multiple root at* $\phi=0$, *it remains a multiple root for all* ϕ,[9]. Next let

$$\tan 2\phi_1=\frac{2\alpha_0}{1-(\alpha_0^2+\beta_0^2)}. \tag{67}$$

We see from eqn (66) that

$$\alpha(\phi)=0 \quad \text{when} \quad \phi=\begin{cases}\phi_1\\ \phi_1\pm\pi/2\\ \phi_1+\pi\end{cases}. \tag{68}$$

We therefore have the result that *unless* $p(\phi)=\pm i$, *there are four orientations of the coordinates, each of them differs by an angle of* $\pi/2$, *for which p is purely imaginary.*

We now denote by θ the angle measured from $\phi=\phi_1$ and write eqn (63) as

$$p(\phi)=p(\phi_1+\theta)=\frac{p(\phi_1)\cos\theta-\sin\theta}{p(\phi_1)\sin\theta+\cos\theta} \tag{69}$$

or

$$\alpha(\phi)+i\beta(\phi)=\frac{\frac{1}{2}(\beta_1^2-1)\sin 2\theta+i\beta_1}{\cos^2\theta+(\beta_1\sin\theta)^2} \tag{70}$$

where

$$\left.\begin{aligned}p(\phi_1)&=i\beta(\phi_1)=i\beta_1\\ p\left(\phi_1+\frac{\pi}{2}\right)&=-\frac{1}{p(\phi_1)}=\frac{i}{\beta_1}.\end{aligned}\right\} \tag{71}$$

Since all p's come in pairs of complex conjugate, we will consider the case $\beta_0>0$. Hence $\beta_1>0$ by eqn (66). Moreover, since ϕ_1 as defined by eqn (67) is not unique, one may choose ϕ_1 such that $\beta_1>1$. We will call β_1 and $1/\beta_1$ the principal values and ϕ_1 and $\phi_1+\pi/2$ the principal directions.

Let

$$\left.\begin{aligned}\gamma &= \tfrac{1}{2}\left(\beta_1 + \frac{1}{\beta_1}\right), & q &= \tfrac{1}{2}\left(\beta_1 - \frac{1}{\beta_1}\right)\\ \beta_1 &= \gamma + q, & \frac{1}{\beta_1} &= \gamma - q.\end{aligned}\right\} \tag{72}$$

Then

$$\gamma^2 - q^2 = 1. \tag{73}$$

The real and imaginary parts of eqn (70) can be written as

$$\alpha(\phi) = \frac{q \sin 2\theta}{\gamma - q\cos 2\theta}, \quad \beta(\phi) = \frac{1}{\gamma - q\cos 2\theta}, \tag{74}$$

or

$$\left.\begin{aligned} q\beta(\phi)\sin 2\theta &= \alpha(\phi)\\ q\beta(\phi)\cos 2\theta &= \gamma\beta(\phi) - 1.\end{aligned}\right\} \tag{75}$$

Elimination of θ between the two equations yields the relation

$$\alpha^2 + (\beta - \gamma)^2 = \gamma^2 - 1 = q^2 \tag{76a}$$

or

$$\alpha^2 + (\beta - \beta_1)\left(\beta - \frac{1}{\beta_1}\right) = 0 \tag{76b}$$

where use has been made of eqns (72) and (73). This is the equation of a circle in which the center of the circle is at $p = \gamma i$ and the radius is q, Fig. 2. The circle passes through the principal values $p = \beta_1 i$ and i/β_1 which correspond to the principal directions $\phi = \phi_1$ and $\phi_1 + \pi/2$, respectively. In view of eqn (13) the circle intersects orthogonally at points t and t' another circle of unit radius with its center at the origin.

We see from eqn (74) that as ϕ varies from 0 to 2π, p traces the circle, eqn (76), clockwise twice. Another circle of the same size but symmetrically placed with respect to the α-axis on the negative β plane is the locus of the complex conjugate of p. As ϕ varies from 0 to 2π, $\bar{p}$ traces this circle counterclockwise twice. Since there are three pairs of complex conjugates for p, we would have three circles each on the positive and negative β plane.

For the circles on the positive β plane, each circle as given by eqn (76b) is determined by the value of β_1. It is clear that *two circles with different β_1 never intersect. Therefore, if p's are distinct at $\phi = 0$, they remain distinct for all ϕ.* The last statement remains valid even if two circles happen to have the same β_1 value.

We stated that each circle is determined by the value β_1. Since β_1, γ and q are all related through eqns (72) and (73), any one of the three will determine the circle. In fact, β_1, γ and q are invariants of p. To find the form of the invariants, we solve γ from eqn (76a) to obtain

$$\gamma = \frac{\alpha^2 + \beta^2 + 1}{2\beta} = \frac{i(p\bar{p} + 1)}{p - \bar{p}}. \tag{77}$$

This relation is invariant to the rotation of the coordinates. The form for the invariant q is

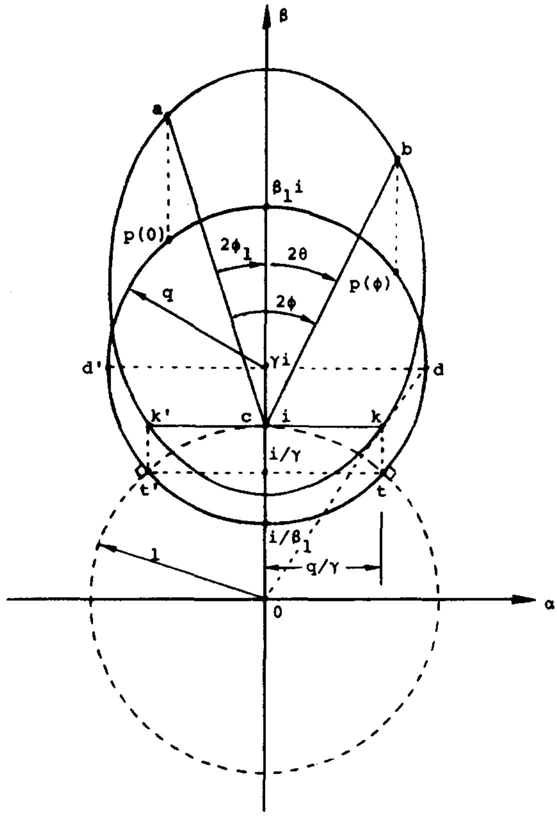

Fig. 2. Graphical solution of $p(\phi)$.

obtained from eqns (73) and (77). We have

$$q^2 = \frac{(\alpha^2+\beta^2-1)^2+4\alpha^2}{4\beta^2} = -\frac{(p^2+1)(\bar{p}^2+1)}{(p-\bar{p})^2}. \tag{78}$$

As to β_1, we find from eqns (72), (77) and (78),

$$\beta_1 = \frac{\alpha^2+\beta^2+1+((\alpha^2+\beta^2-1)^2+4\alpha^2)^{1/2}}{2\beta}. \tag{79}$$

For isotropic materials, $\gamma = \beta_1 = 1$ and $q = 0$. Thus instead of a circle we have a point located at $p = i$.

It is clear from Fig. 2 that the absolute value of the real part of p can never exceed the value q. To find the orientation at which the real part of p reaches $\pm q$, we use eqns (74):

$$\frac{q \sin 2\theta}{\gamma - q \cos 2\theta} = \pm q. \tag{80}$$

Making use of eqn (73) and solving for θ, one obtains

$$\sin 2\theta = \pm 1/\gamma. \tag{81}$$

We therefore have the result that

$$p = \pm q + \gamma i \quad \text{at} \quad \phi = \phi_1 \pm \tfrac{1}{2}\sin^{-1}(1/\gamma). \tag{82}$$

These are points d and d' in Fig. 2.

When the material is symmetric with respect to the (x_2, x_3) plane or the the (x_1, x_3) plane, the sectic equation for p reduces to a cubic in p^2. At least one of the p^2 roots is real and the other two may be real or complex conjugates. If they are complex conjugates, they can be written as, after taking the square root,

$$p = \pm\alpha_0 + i\beta_0, \quad \pm\alpha_0 - i\beta_0. \tag{83}$$

The first two roots have the same γ value according to eqn (77) and the two associated circles coincide. Therefore, we have at most two circles instead of three on the upper and lower plane.

While the circle in Fig. 2 is the locus of p as ϕ varies from 0 to 2π, one could obtain graphically the angle ϕ associated with each point on the circle. To this end, we draw an ellipse given by

$$\alpha^2 + \left(\frac{\beta}{\gamma} - \gamma\right)^2 = q^2. \tag{84}$$

The lengths of the major and minor axes of the ellipse are γq and q, respectively. Moreover, the point $p = i$ which is identified by c in Fig. 2 is one of the foci and the α-axis is the directrix. The distance between the foci is $2q^2$. Hence the eccentricity e of the ellipse is

$$e = q/\gamma = (1 - \gamma^{-2})^{1/2}. \tag{85}$$

Comparing eqn (84) with (76a), we see that if we stretch the p-plane uniformly in the β-direction with the stretch ratio γ while holding the α-axis fixed, the circle of eqn (76a) becomes the ellipse of eqn (84). Thus there is a one-to-one mapping between points on the circle and points on the ellipse. For instance, points $p(0)$, $p(\phi)$, t and t' on the circle correspond to points a, b, k and k', respectively, on the ellipse. To find the angle ϕ between $p(0)$ and $p(\phi)$ on the circle, we connect the corresponding points a and b on the ellipse to the point c. The angle between ca and cb is 2ϕ.

We first show that the angle between ca and the β-axis is $2\phi_1$. Since the coordinates of point a are $(\alpha_0, \gamma\beta_0)$, we see from Fig. 2 that

$$\tan 2\phi_1 = \frac{-\alpha_0}{\gamma\beta_0 - 1}. \tag{86}$$

This is identical to eqn (67) if γ of eqn (77) is used in eqn (86).

Next we show that the angle between cb and the β-axis is 2θ. Noticing that the coordinates of point b are $(\alpha(\phi), \gamma\beta(\phi))$, the length of the line cb is

$$\overline{cb} = \{\alpha^2(\phi) + [\gamma\beta(\phi) - 1]^2\}^{1/2} = q\beta(\phi) \tag{87}$$

by eqn (75). From Fig. 2, it is seen that

$$\left.\begin{aligned} \overline{cb}\sin 2\theta &= \alpha(\phi) \\ \overline{cb}\cos 2\theta &= \gamma\beta(\phi) - 1. \end{aligned}\right\} \tag{88}$$

This is identical to eqn (75) in view of eqn (87).

It can be shown that t and t' in Fig. 2 are associated with $\phi = \phi_1 \pm \pi/4$ and

$$p(\phi_1 \pm \pi/4) = \pm e + i/\gamma. \tag{89}$$

Therefore, the abcissa of point t provides the eccentricity e of the ellipse. Since $\theta = \pi/4$, the lines connecting point c to the corresponding points k and k' on the ellipse are at the right angle with the β-axis. The line $0k$ is tangential to the ellipse at k, and meets point d if extended.

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